Braess' Paradox in City Planning

Yang Wu

March 2020

Contents

1 Task I

1.1 Basic idea

First, we need to establish some notations:

- $t_i(\phi)$ represents the time it takes to traverse road i with ϕ drivers on it
- $|\Phi|$ represents the total number of drivers. This total number of drivers was then split into potentially different quantities of drivers taking different paths. This was written $\Phi_{a_1 a_2 \ldots a_n}$ to represent the number of drivers on the path from a_1 to a_2 to . . . to a_n
- The expression $T(\Phi)$ denoted the maximum travel time for a given distribution of drivers. That is, $T(\Phi)$ represented the worst-case travel time among all the drivers

Then, we have the following model:

$$t_1(\phi) = t_3(\phi) = 10\phi$$
$$t_2(\phi) = t_4(\phi) = 50 + \phi$$
$$t_5(\phi) = 10 + \phi$$

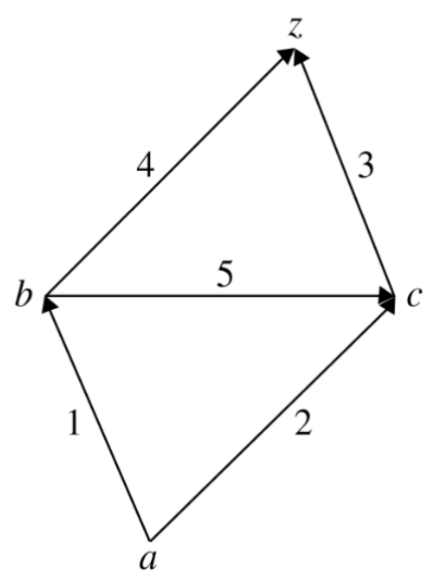

Figure 1: Network

2 Task II

2.1 Verify Braess' claims

a) Given a total flow of $|\Phi| = 2$ is to be guided from a to z, the optimal solution is:

$$\Phi_{abcz} = 2, \quad \Phi_{abz} = \Phi_{acz} = 0, \quad T(\Phi) = 52$$

$T(\Phi_{abcz})$ can be computed as follows:

$$t_1(2) = 10 \times 2 = 20$$
$$t_3(2) = 10 \times 2 = 20$$
$$t_5(2) = 10 + 2 = 12$$

Therefore:

$$\begin{aligned} T(\Phi_{abcz}) &= t_1(2) + t_5(2) + t_3(2) \\ &= 20 + 12 + 20 \\ &= 52 \end{aligned}$$

This is the maximum travel time for path **abcz** and it is the optimal solution. Any redistribution will cause the maximum travel time to increase. According to Braess' paper, "[t]he time that is needed to reach the destination in the most unfavorable case measures how well the flows are distributed." For a proof that any redistribution would increase maximum travel time, we examine the following solution:

$$\Phi_{abcz} = 0, \quad \Phi_{abz} = \Phi_{acz} = 1, \quad T(\Phi) = ?$$

$T(\Phi_{abz})$ and $T(\Phi_{acz})$ can be computed as follows:

$$\begin{aligned} t_1(1) &= 10 \times 1 = 10 \\ t_2(1) &= 50 + 1 = 51 \\ t_3(1) &= 10 \times 1 = 10 \\ t_4(1) &= 50 + 1 = 51 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abz}) &= t_1(1) + t_4(1) \\ &= 10 + 51 \\ &= 61 \end{aligned}$$

$$\begin{aligned} T(\Phi_{acz}) &= t_2(1) + t_3(1) \\ &= 51 + 10 \\ &= 61 \end{aligned}$$

Notice that these travel times exceed the optimal solution $T(\Phi) = 52$. Having all drivers ($flow = 2$) take the path **abcz** is optimal since it minimizes the maximum travel time among all drivers. For links 2 and 4 in Figure 1, travel time is 50 with no cars on the road, indicating that these are relatively longer paths compared to links 1 and 3. For a total flow of 2, it would be optimal to avoid these longer paths in favor of the shorter path **abcz**.

b) Given a total flow of $|\Phi| = 6$ is to be guided from a to z, the optimal solution is:

$$\Phi_{abcz} = 0, \quad \Phi_{abz} = \Phi_{acz} = 3, \quad T(\Phi) = 83$$

$T(\Phi_{abz})$ and $T(\Phi_{acz})$ can be computed as follows:

$$\begin{aligned} t_1(3) &= 10 \times 3 = 30 \\ t_2(3) &= 50 + 3 = 53 \\ t_3(3) &= 10 \times 3 = 30 \\ t_4(3) &= 50 + 3 = 53 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abz}) &= t_1(3) + t_4(3) \\ &= 30 + 53 \\ &= 83 \end{aligned}$$

$$\begin{aligned} T(\Phi_{acz}) &= t_2(3) + t_3(3) \\ &= 53 + 30 \\ &= 83 \end{aligned}$$

If the total flow is 6, the optimal arrangement splits up the drivers to take paths **abz** and **acz** (a sub-total flow of 3 on each path). If we redistribute, the maximum travel time will increase. For instance:[1]

$$\Phi_{abcz} = 0, \quad \Phi_{abz} = 4, \quad \Phi_{acz} = 2, \quad T(\Phi) = ?$$

$T(\Phi_{abz})$ and $T(\Phi_{acz})$ can be computed as follows:

$$\begin{aligned} t_1(4) &= 10 \times 4 = 40 \\ t_2(2) &= 50 + 2 = 52 \\ t_3(2) &= 10 \times 2 = 20 \\ t_4(4) &= 50 + 4 = 54 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abz}) &= t_1(4) + t_4(4) \\ &= 40 + 54 \\ &= 94 \end{aligned}$$

$$\begin{aligned} T(\Phi_{acz}) &= t_2(2) + t_3(2) \\ &= 52 + 20 \\ &= 72 \end{aligned}$$

Here, the worst-case travel time among all drivers is $T(\Phi_{abz}) = 94$, which exceeds the optimal travel time of 83. As to why a redistribution would increase the maximum travel time, notice that, in Figure 1, the two paths **abz** and **acz** are the same in terms of distance traveled. Holding distance constant, the path with a larger flow would yield a higher maximum travel time. Having unequal numbers of drivers on two equal paths would have the following effect: since travel time on path **acz** is shorter, 72, there is an incentive for drivers on path **abz** to switch. The switching will take place until a nash-equilibrium is reached on paths **abz** and **acz**.

c) Given a total flow of $|\Phi| = 20$ is to be guided from a to z, the optimal solution is:

$$\Phi_{abcz} = 0, \quad \Phi_{abz} = \Phi_{acz} = 10, \quad T(\Phi) = 160$$

$T(\Phi_{abz})$ and $T(\Phi_{acz})$ can be computed as follows:

$$\begin{aligned} t_1(10) &= 10 \times 10 = 100 \\ t_2(10) &= 50 + 10 = 60 \\ t_3(10) &= 10 \times 10 = 100 \\ t_4(10) &= 50 + 10 = 60 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abz}) &= t_1(10) + t_4(10) \\ &= 100 + 60 \\ &= 160 \end{aligned}$$

[1] Note that we only present one of the many possibilities. Different cases can be used to demonstrate the idea that a deviation from the optimal arrangement would increase maximum travel time.

$$\begin{aligned} T(\Phi_{acz}) &= t_2(10) + t_3(10) \\ &= 60 + 100 \\ &= 160 \end{aligned}$$

Again, if the total flow is 20, the optimal arrangement splits up the drivers to take paths **abz** and **acz** (sub-total of 10 on each path). Redistributing the drivers would increase the maximum travel time. For proof:

$$\Phi_{abcz} = 0, \quad \Phi_{abz} = 5, \quad \Phi_{acz} = 15, \quad T(\Phi) = ?$$

$T(\Phi_{abz})$ and $T(\Phi_{acz})$ can be computed as follows:

$$\begin{aligned} t_1(5) &= 10 \times 5 = 50 \\ t_2(15) &= 50 + 15 = 65 \\ t_3(15) &= 10 \times 15 = 150 \\ t_4(5) &= 50 + 5 = 55 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abz}) &= t_1(5) + t_4(5) \\ &= 50 + 55 \\ &= 105 \end{aligned}$$

$$\begin{aligned} T(\Phi_{acz}) &= t_2(15) + t_3(15) \\ &= 65 + 150 \\ &= 215 \end{aligned}$$

Here, the worst-case travel time among all the drivers is $T(\Phi_{acz}) = 215$, which exceeds the optimal maximum travel time of 160. The same idea from case b) applies here. Since the two paths **abz** and **acz** cover the same distance, the path with more drivers would output a larger maximum travel time. Again, the differing travel times would lead to switching among drivers who are on the more congested path (in our case, path **acz**), eventually leading our network to a nash-equilibrium.

3 Task III

3.1 Modeling four scenarios using linear equations

a) 1 km of three-lane interstate highway: The intercept "b" indicates the travel time with no cars on the highway, and is likely to be small positive number. We would use the following formula to estimate "b":

$$Time = \frac{Distance}{Speed}$$

The typical freeway in the state of New Jersey has a speed limit of 89 km/h. Therefore, for a 1 km three-lane interstate highway, the intercept "b" can be estimated as follows:

$$\frac{1km}{89km/h} \approx 0.0112 \; hours \approx 40 \; seconds$$

Next, as the number of cars increases, travel time will increase since the highway becomes more congested. The slope "m" can be interpreted as follows: the amount of change to travel time due to the number of cars increasing by 1; thus, "m" should be positive. For a three-lane highway with a speed limit of 89 km/h, traffic is not expected to get extremely congested. In other words, the number of cars is not expected to increase travel time by a substantial amount until it becomes fairly large, say 1000. Given this assumption, a reasonable estimate of "m" could be 1.

1 km of three-lane interstate highway that shrinks from three lanes to two lanes halfway through: Since the distance remains the same, the intercept "b" is expected to be the same.[2] However, the highway shrinks to two lanes halfway through, likely leading to more congestion. In addition, as the number of lanes is reduced, drivers on the third lane would have to slow down for merging, leading to longer travel times. Thus, as the number of cars increases, travel can be prolonged further. It is reasonable to expect the slope "m" in this case to be larger than that of the last scenario. In this case, the number of cars will have a larger impact on travel time, i.e. a larger slope "m," given the specified condition that the highway shrinks to two-lanes. Therefore, a reasonable estimate of "m" could be 2.

A toll booth at the entrance to a tunnel: The intercept "b" is zero; we argue that, if there are no cars at the toll booth, travel time is so minuscule that it can safely assumed to be zero. For the slope "m", we argue that it should be positive and its magnitude should be greater than those of the slopes in both scenarios above. Typically, the law requires drivers to slow down (65-80 km per hour) before entering a tunnel regardless of whether there is a toll booth; this is usually to prepare for the narrowing of the road. A toll booth at the entrance to a tunnel is expected to slow down traffic further (30-80 km per hour). We argue that the number of cars at the toll booth would have a substantial impact on travel time where a larger number of cars would lead to longer lines, which, in turn, would increase travel time. Keeping with this assumption, a reasonable estimate of "m" could be 5.

The stretch of the tunnel following the toll booth: In the United States, the NFPA[3] definition of a tunnel is "[a]n underground structure with a design length greater than [0.023 km]..." The currently longest tunnel is the State Route 99 tunnel, which stretches 2.83 km in length. We assume that a typical tunnel has a length somewhere in the middle between these extrema:

$$\frac{2.83miles + 0.023mile}{2} \approx 1.4265\ km$$

Using this distance, the intercept "b" can be estimated as follows[4]:

$$\frac{1.42km}{80km/h} \approx 0.0178\ hours \approx 64\ seconds$$

The slope "m" is expected to be positive since more cars means more congestion, increasing travel time. In addition, the Port Authority of NY & NJ recommends that vehicles "operating at a maximum speed of 35 miles per hour shall maintain a distance of at least 75 feet behind the vehicle immediately preceding it in the same lane." This regulation is expected to further prolong travel in a tunnel as the number of cars increases. The magnitude of the slope in this case should be the largest among all four scenarios. All things considered, a reasonable estimate of the slope "m" could be 10.

3.2 Road map for Braess' example network

b) A road map can be constructed to represent Braess' example network. In Figure 2, the distances of the links reflect the linear equations given. We assume the units of time to be seconds and the speed limit to be 89 km/h. Then, the distances are computed according to the intercept values of 50, 10, and 0:

$$t_1(\phi) = t_3(\phi) = 10\phi$$
$$t_2(\phi) = t_4(\phi) = 50 + \phi$$
$$t_2(\phi) = 10 + \phi$$

[2] We continue to make the same assumption with regards to speed limit.
[3] National Fire Protection Association
[4] We assume the speed limit in tunnels to be 80 km/h, which is not unreasonable for a state like New Jersey

The equations for links 1 and 3 have intercept value of 0. Notice that for links 1 and 3, we assume the intercept "b" is 6 seconds rather than 0 second. Here, the assumption is that a 6-second period is so minuscule that it can be assumed away in the equation.

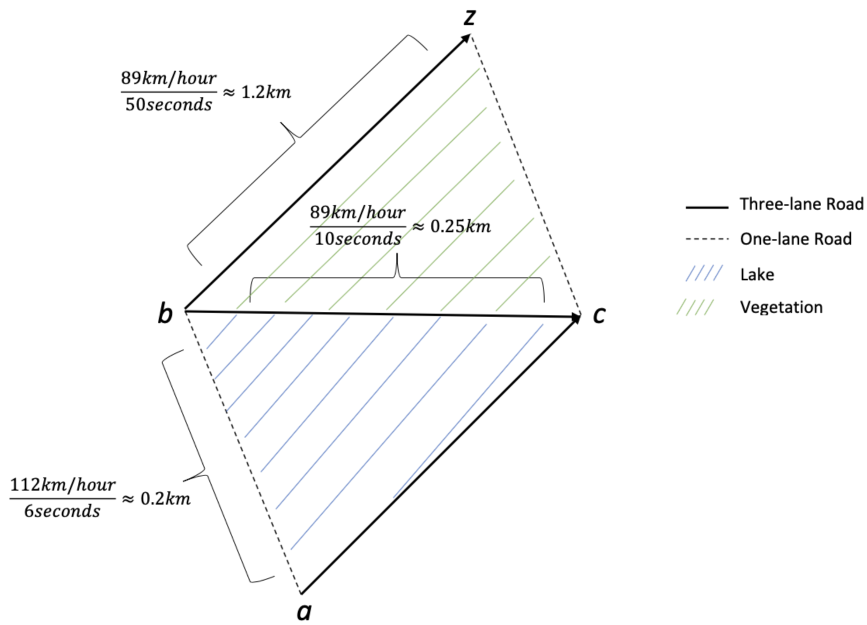

Figure 2: Road Map

In Figure 2, we reflect the differences in the slopes of the equations using three-lane and one-lane roads, which are represented by solid and dashed lines, respectively. For links 2,4, and 5, the slope is 1; keeping with our assumptions earlier, this indicates three-lane roads. Links 1 and 3 indicate one-lane roads, since their slopes are greater in magnitude. Due to narrower roads, travel time is more sensitive to increases in the number of cars. Lastly, we also include labels for geographical features such as lake and vegetation.

4 Task IV

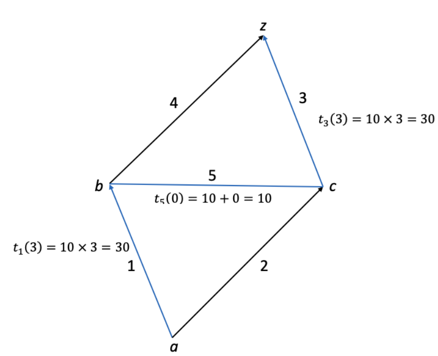

Figure 3: Diagram with link travel times

In case (b), the optimal flow moves along paths **abz** and **acz**. The links that are occupied are links 1,2,3, and 4. Notice that there are no cars driving on link 5. Therefore, while all drivers are driving along paths **abz** and **acz** as indicated by the optimal arrangement, there exists an alternative path **abcz** (highlighted in blue in Figure 3) such that the travel time is:

$$\begin{aligned} T(\Phi_{abcz}) &= t_1(3) + t_5(0) + t_3(3) \\ &= 30 + 10 + 30 \\ &= 70 \end{aligned}$$

And $T(\Phi_{abcz}) = 70 < 83 = |T(\Phi)|$. Suppose one driver on path **acz** realizes this and switches to take path **abcz**. The network changes as follows:

$$\Phi_{abcz} = 1, \quad \Phi_{abz} = 3, \quad \Phi_{acz} = 2, \quad T(\Phi) = ?$$

$T(\Phi_{abz})$ and $T(\Phi_{acz})$ can be computed as follows:

$$\begin{aligned} t_1(4) &= 10 \times 4 = 40 \\ t_2(2) &= 50 + 2 = 52 \\ t_3(3) &= 10 \times 3 = 30 \\ t_4(3) &= 50 + 3 = 53 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abz}) &= t_1(4) + t_4(3) \\ &= 40 + 53 \\ &= 93 \end{aligned}$$

$$\begin{aligned} T(\Phi_{acz}) &= t_2(2) + t_3(3) \\ &= 52 + 30 \\ &= 82 \end{aligned}$$

$T(\Phi_{abcz})$ can be computed as follows:

$$\begin{aligned} t_1(4) &= 10 \times 4 = 40 \\ t_3(3) &= 10 \times 3 = 30 \\ t_5(1) &= 10 + 1 = 11 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abcz}) &= t_1(4) + t_5(1) + t_3(3) \\ &= 40 + 11 + 30 \\ &= 81 \end{aligned}$$

For drivers on path **acz**, the new travel time is 82 as one driver switches to take path **abcz**, which has the shortest travel time, 81. However, the maximum travel time among all drivers is now 93 for drivers on path **abz**. Now, there is an incentive for these drivers to switch to other paths. Let's examine a scenario where a driver from path **abz** decides to switch to path **abcz**:

$$\Phi_{abcz} = 2, \quad \Phi_{abz} = 2, \quad \Phi_{acz} = 2, \quad T(\Phi) = ?$$

$T(\Phi_{abz})$ and $T(\Phi_{acz})$ can be computed as follows:

$$\begin{aligned} t_1(4) &= 10 \times 4 = 40 \\ t_2(2) &= 50 + 2 = 52 \\ t_3(4) &= 10 \times 4 = 40 \\ t_4(2) &= 50 + 2 = 52 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abz}) &= t_1(4) + t_4(2) \\ &= 40 + 52 \\ &= 92 \end{aligned}$$

$$\begin{aligned} T(\Phi_{acz}) &= t_2(2) + t_3(4) \\ &= 52 + 40 \\ &= 92 \end{aligned}$$

$T(\Phi_{abcz})$ can be computed as follows:

$$\begin{aligned} t_1(4) &= 10 \times 4 = 40 \\ t_3(4) &= 10 \times 2 = 40 \\ t_5(2) &= 10 + 2 = 12 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abcz}) &= t_1(4) + t_5(2) + t_3(4) \\ &= 40 + 12 + 40 \\ &= 92 \end{aligned}$$

For drivers on all three paths, the new travel time is 92. We have reached a new nash-equilibrium as there is no incentive among drivers to deviate from the current state. Notice, however, that the new maximum travel time is longer than that of under the optimal flow: $T(\Phi_{new}) = 92 > 83 = |T(\Phi_{optimal})|$. Lastly, suppose all drivers under the optimal flow realize the time-saving alternative path and everyone switches to take path **abcz**. The new maximum travel time is:

$$\begin{aligned} t_1(6) &= 10 \times 6 = 60 \\ t_3(6) &= 10 \times 6 = 60 \\ t_5(6) &= 10 + 6 = 16 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abcz}) &= t_1(6) + t_5(6) + t_3(6) \\ &= 60 + 16 + 60 \\ &= 136 \end{aligned}$$

Evidently, $T(\Phi_{abcz}) = 136 > 83 = |T(\Phi_{optimal})|$. As all drives switch to take the alternative path, the path gets very congested, increasing the maximum travel time.

5 Task V

5.1 Removing links for case a) from Task I & II

Removal of link 1: Having removed link 1, the only viable path remaining is path **acz**. As a result, all drivers would switch to this path. The new maximum travel time can be computed as follows:

$$\begin{aligned} t_2(2) &= 50 + 2 = 52 \\ t_3(2) &= 10 \times 2 = 20 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{acz}) &= t_2(2) + t_3(2) \\ &= 52 + 20 \\ &= 72 \end{aligned}$$

Evidently, $T(\Phi_{acz}) = 72 > 52 = |T(\Phi_{optimal})|$.

Removal of link 2: Having removed link 2, there are two remaining paths **abz** and **abcz**. We examine the scenario where one driver chooses path **abz** and the other chooses **abcz**. The new travel times can be computed as follows:

$$\begin{aligned} t_1(2) &= 10 \times 2 = 20 \\ t_4(1) &= 50 + 1 = 51 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abz}) &= t_1(2) + t_4(1) \\ &= 20 + 51 \\ &= 71 \end{aligned}$$

$T(\Phi_{abcz})$ can be computed as follows:

$$\begin{aligned} t_1(2) &= 10 \times 2 = 20 \\ t_3(1) &= 10 \times 1 = 10 \\ t_5(1) &= 10 + 1 = 11 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abcz}) &= t_1(2) + t_5(1) + t_3(1) \\ &= 20 + 11 + 10 \\ &= 41 \end{aligned}$$

Again, the new worst-case travel time among all drivers is $T(\Phi_{abz}) = 71 > 52 = |T(\Phi_{optimal})|$. We will not compute the travel times for the scenario where all drivers choose path **abz** or **abcz**. We know that if having a total flow of 1 on each path does not improve traffic flow, having a total flow of 2 on these same paths certainly would not improve traffic flow.

Removal of link 3: Having removed link 3, the only viable path remaining is path **abz**. As a result, all drivers would switch to this path. The new maximum travel time can be computed as follows:

$$\begin{aligned} t_1(2) &= 10 \times 2 = 20 \\ t_4(2) &= 50 + 2 = 52 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abz}) &= t_1(2) + t_4(2) \\ &= 20 + 52 \\ &= 72 \end{aligned}$$

Here, $T(\Phi_{abz}) = 72 > 52 = |T(\Phi_{optimal})|$.

Removal of link 4: Having removed link 4, there are two remaining paths **acz** and **abcz**. Again, we examine the scenario where one driver chooses path **acz** and the other chooses **abcz**. The new travel times can be computed as follows:

$$\begin{aligned} t_2(1) &= 50 + 1 = 51 \\ t_3(2) &= 10 \times 2 = 20 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{acz}) &= t_2(1) + t_3(2) \\ &= 51 + 20 \\ &= 71 \end{aligned}$$

$T(\Phi_{abcz})$ can be computed as follows:

$$\begin{aligned} t_1(1) &= 10 \times 1 = 10 \\ t_3(2) &= 10 \times 2 = 20 \\ t_5(1) &= 10 + 1 = 11 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abcz}) &= t_1(1) + t_5(1) + t_3(2) \\ &= 10 + 11 + 20 \\ &= 41 \end{aligned}$$

Again, the new worst-case travel time among all drivers is $T(\Phi_{acz}) = 71 > 52 = |T(\Phi_{optimal})|$. Furthermore, we know that if having a total flow of 1 on each path does not improve traffic flow, having a total flow of 2 on these same paths certainly would not improve traffic flow.

Removal of link 5: Removing link 5 ensures that drivers either have to take path **abz** or **acz**. We examine the scenario where one driver chooses path **abz** and the other chooses **acz**. The new maximum travel times can be computed as follows:

$$\begin{aligned} t_1(1) &= 10 \times 1 = 10 \\ t_4(1) &= 50 + 1 = 51 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abz}) &= t_1(1) + t_4(1) \\ &= 10 + 51 \\ &= 61 \end{aligned}$$

Here, $T(\Phi_{abz}) = 61 > 52 = |T(\Phi_{optimal})|$. $T(\Phi_{acz})$ can be computed as follows:

$$\begin{aligned} t_2(1) &= 50 + 1 = 51 \\ t_3(1) &= 10 \times 1 = 10 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{acz}) &= t_2(1) + t_3(1) \\ &= 51 + 10 \\ &= 61 \end{aligned}$$

We have verified that $T(\Phi_{acz}) = 61 > 52 = |T(\Phi_{optimal})|$. We also know that a total flow of 2 on either one of these paths would increase travel time even more. For case a), we have confirmed that no removal of a link will result in improved flow.

5.2 Removing links for case c) from Task I & II

Removal of link 1: Having removed link 1, the only viable path remaining is path **acz**. As a result, the total flow of 20 would switch to this path. The new maximum travel time can be computed as follows:

$$t_2(20) = 50 + 20 = 70$$
$$t_3(20) = 10 \times 20 = 200$$

Therefore:

$$\begin{aligned} T(\Phi_{acz}) &= t_2(20) + t_3(20) \\ &= 70 + 200 \\ &= 270 \end{aligned}$$

As can been seen, we observe a huge increase in travel time as all drivers are forced to take path **acz**. $T(\Phi_{acz}) = 270 > 160 = |T(\Phi_{optimal})|$.

Removal of link 2: Again, removing link 2 leaves two remaining paths **abz** and **abcz**. To avoid computing travel times for all combinations of distribution, we will examine the scenario where half of the drivers choose path **abz** and the other half choose path **abcz**, i.e., a sub-total flow of 10 on each. The new travel times can be computed as follows:

$$t_1(20) = 10 \times 20 = 200$$
$$t_4(10) = 50 + 10 = 60$$

Therefore:

$$\begin{aligned} T(\Phi_{abz}) &= t_1(20) + t_4(10) \\ &= 200 + 60 \\ &= 260 \end{aligned}$$

$T(\Phi_{abcz})$ can be computed as follows:

$$t_1(20) = 10 \times 20 = 200$$
$$t_3(10) = 10 \times 10 = 100$$
$$t_5(10) = 10 + 10 = 20$$

Therefore:

$$\begin{aligned} T(\Phi_{abcz}) &= t_1(20) + t_5(10) + t_3(10) \\ &= 200 + 20 + 100 \\ &= 320 \end{aligned}$$

The new worst-case travel time among all drivers has doubled, $T(\Phi_{abcz}) = 320 > 160 = |T(\Phi_{optimal})|$. We also know that any redistribution will only further increase the worst-case travel time among all drivers. This is because, as drivers on path **abcz** switch to path **abz** in hopes of shortening travel time, travel time on **abz** will only increase further. And, even with a sub-total flow of 10, $T(\Phi_{abz}) = 260 > 160 = |T(\Phi_{optimal})|$ and $T(\Phi_{abcz}) = 320 > 160 = |T(\Phi_{optimal})|$. Therefore, we do not need to compute travels time for all combinations of distribution; removing link 2 does not improve traffic flow.

Removal of link 3: Removing link 3 leaves only path **abz**. The new maximum travel time can be computed as follows:

$$t_1(20) = 10 \times 20 = 200$$
$$t_4(20) = 50 + 20 = 70$$

Therefore:

$$\begin{aligned} T(\Phi_{abz}) &= t_1(20) + t_4(20) \\ &= 200 + 70 \\ &= 270 \end{aligned}$$

Removing link 3 results in the same situation when we remove link 1, $T(\Phi_{abz}) = 270 > 160 = |T(\Phi_{optimal})|$.

Removal of link 4: Removing link 4 leaves two remaining paths **acz** and **abcz**. Again, we examine the scenario where a sub-total flow of 10 chooses path **acz** and the other 10 choose **abcz**. The new travel times can be computed as follows:

$$\begin{aligned} t_2(10) &= 50 + 10 = 60 \\ t_3(20) &= 10 \times 20 = 200 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{acz}) &= t_2(10) + t_3(20) \\ &= 60 + 200 \\ &= 260 \end{aligned}$$

$T(\Phi_{abcz})$ can be computed as follows:

$$\begin{aligned} t_1(10) &= 10 \times 10 = 100 \\ t_3(20) &= 10 \times 2 = 200 \\ t_5(10) &= 10 + 10 = 20 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abcz}) &= t_1(10) + t_5(10) + t_3(20) \\ &= 100 + 20 + 200 \\ &= 320 \end{aligned}$$

The result is the same as removing link 2, the new worst-case travel time among all drivers is $T(\Phi_{abcz}) = 320 > 160 = |T(\Phi_{optimal})|$.

Removal of link 5: Removing link 5 forces drivers to take either path **abz** or **acz**. We examine the scenario where a sub-total of 10 chooses path **abz** and the others choose **acz**. Notice also that this follows the optimal arrangement. Therefore, the same argument with regards to redistribution applies here. A redistribution will increase the maximum travel time among all drivers. The new maximum travel times can be computed as follows:

$$\begin{aligned} t_1(10) &= 10 \times 10 = 100 \\ t_4(10) &= 50 + 10 = 60 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abz}) &= t_1(10) + t_4(10) \\ &= 100 + 60 \\ &= 160 \end{aligned}$$

Here, $T(\Phi_{abz}) = 160 = 160 = |T(\Phi_{optimal})|$. $T(\Phi_{acz})$ can be computed as follows:

$$t_2(10) = 50 + 0 = 60$$
$$t_3(10) = 10 \times 10 = 100$$

Therefore:

$$\begin{aligned} T(\Phi_{acz}) &= t_2(10) + t_3(10) \\ &= 60 + 100 \\ &= 160 \end{aligned}$$

Now, we have verified that $T(\Phi_{acz}) = 160 = 160 = |T(\Phi_{optimal})|$. Removing link 5 does not improve flow; it simply ensures that drivers' behaviors can be expected to follow the optimal solution, assuming that drivers are rational agents who wish to minimize travel time. The network is in a nash-equilibrium and no individual driver has any incentive to deviate from his or her current path. For case c), we have confirmed that no removal of a link will result in improved traffic flow.

6 Task VI

6.1 Alternative measures for assessing traffic flow quality

The two different measures that Braess used to determine the quality of a particular traffic flow are:

- the maximum, i.e., the worst-case travel time among all drivers, $|T(\Phi)|$
- the mean value of the travel time, $\frac{1}{|\Phi|}\sum\limits_{\beta} \Phi_\beta T_\beta(\Phi)$

One situation where the maximum travel time is more appropriate for accessing quality of traffic flow could be a trip from a nursing home or an elementary school to the nearest hospital. Traffic planners would wish to minimize travel time for these trips by controlling for the worst-case travel time. The idea is that, if the worst-case travel time is reasonably short, then the quality of traffic flow is optimal.

One possible situation where the mean value of the travel time is arguably more appropriate could be that there are many paths, β, for a given origin-destination pair. It may be that the maximum travel time for this origin-destination pair is an outlier.[5] In this case, using the maximum travel time as a measure to determine the quality of traffic flow may not tell the whole story. In other words, using the maximum travel time would make the traffic flow seem worse than it really is for the average driver. It may be that the assumption that all drivers are rational agents who wish to minimize travel time is not the reality, and some drivers choose the path that yields the maximum travel time for reasons unknown to the researcher (e.g. the views are better or fewer speeding cameras or toll booths). In this case, using a measure of central tendency may be preferred over using the maximum travel time. An important property of the mean is that it includes every path's travel time,$T_\beta(\Phi)$, as part of the calculation. This measure paints a relatively more *complete* picture of the quality of traffic flow given a origin-destination pair.

To take this one step further, we may even consider using a more robust measure of central tendency like the median. Depending on the distribution of travel times and the number of paths, we could even explore some non-parametric alternatives for measuring central tendency, e.g., the Hodges–Lehmann statistic. The field of statistics offers a great variety of alternative measures to solve the problem of assessing the quality of a particular traffic flow under varying conditions and assumptions.

[5]Here, we borrow a concept from statistics, an outlier is a data point that differs significantly from the other observations. In our case, this means that the maximum travel time in a sample may be much longer than the other travel times for a given origin-destination pair due to some factors other than physical distance.

7 Task VII

7.1 Declaring functions and variables

We wish to see what the optimal flow looks like when we analyze the travel times using average rather than maximum travel time.

a) We declare variables as follows:

- Let x represent the number of drivers who use path **abz**.
- Let y represent the number of drivers who use path **acz**.
- Let z represent the number of drivers who use path **abcz**.

We only need to consider these three paths since the network comprises only one-way connections. The direction of the arrows determines that there are only three distinct paths from **origin** a to **destination** z.

b) We express travel time as a function of several real variables, x, y, and z. Travel time for path **abz** can be expressed as follows:

$$T_{abz}(x, y, z) = 10(x + z) + (50 + x) \tag{1}$$

Here, the first term $10(x + z)$ of equation 1 represents the travel time for link 1 from nodes **a** to **b**. To calculate travel time, we must take into account both the number of drivers traveling along path **abz** and **abcz** as both groups travel on link 1. This term involves both the variable **x** and **z**. The second term $(50+x)$ is simply the linear equation that outputs the travel time for link 4 from nodes **b** to **z**. Lastly, the function can be further simplified as follows:

$$\begin{aligned} T_{abz}(x, y, z) &= 10(x + z) + (50 + x) \\ &= 10x + 10z + 50 + x \\ &= 11x + 10z + 50 \end{aligned}$$

Travel time for path **acz** can be expressed as follows:

$$T_{acz}(x, y, z) = (50 + y) + 10(y + z) \tag{2}$$

The first term $(50 + y)$ of equation 2 represents the travel time for link 2 from nodes **a** to **c**. This only involves the variable **y** since we only need to consider the number of drivers traveling along path **acz**. The second term $10(y + z)$ represents the travel time for link 3 from nodes **c** to **z**. Again, for this link, we must take into account both the number of drivers traveling along path **acz** and **abcz** as both groups travel on link 3. This term involves both the variable **y** and **z**. Lastly, the function can be further simplified as follows:

$$\begin{aligned} T_{acz}(x, y, z) &= (50 + y) + 10(y + z) \\ &= 50 + y + 10y + 10z \\ &= 11y + 10z + 50 \end{aligned}$$

Travel time for path **abcz** can be expressed as follows:

$$T_{abcz}(x, y, z) = 10(x + z) + (10 + z) + 10(y + z) \tag{3}$$

The first and third term of equation 3 represent travel times for link 1 from nodes **a** to **b** and link 3 from nodes **c** to **z**. These terms involve two variables for the same reasons described earlier. The second term $(10+z)$ is simply the equation that outputs the travel time for link 5 from nodes **b** to **c**. Lastly, the function can be further simplified as follows:

$$\begin{aligned} T_{abcz}(x, y, z) &= 10(x + z) + (10 + z) + 10(y + z) \\ &= 10x + 10z + 10 + z + 10y + 10z \\ &= 10x + 10y + 21z + 10 \end{aligned}$$

c) The average travel time is given by the following formula:

$$A(x,y,z) = \frac{x \cdot T_{abz}(x,y,z) + y \cdot T_{acz}(x,y,z) + z \cdot T_{abcz}(x,y,z)}{2} \tag{4}$$

The average is simply the sum of all travel times divided by the total number of drivers. The numerator of equation 4 finds the sum of all travel times for the given origin-destination pair **az**. Specifically, there would be x number of drivers traveling on path **abz** whose travel times are represented by $T_{abz}(x,y,z)$; there would be y number of drivers traveling on path **acz** whose travel times are represented by $T_{acz}(x,y,z)$; lastly, there would be z number of drivers traveling on path **abcz** whose travel times are represented by $T_{abcz}(x,y,z)$. The denominator of equation 4 is the total flow, which is 2 in case a).

7.2 Optimization

d) The total flow in case a) is 2. Therefore, the sum of the three variables representing the number of drivers on each of the three paths, $x+y+z$, cannot exceed the total flow. In other words, we wish to minimize the mean value of travel times *given a total flow of 2 drivers.* The objective function is:

$$A(x,y,z) = \frac{x \cdot T_{abz}(x,y,z) + y \cdot T_{acz}(x,y,z) + z \cdot T_{abcz}(x,y,z)}{2}$$

and it is subject to the constraint:

$$g(x,y,z) = x + y + z = 2 \tag{5}$$

e) To minimize $A(x,y,z)$ subject to the constraint $g(x,y,z) = C$, we use the Lagrangian function:

$$\mathcal{L}(x,y,z,\lambda) = A(x,y,z) - \lambda(g(x,y,z) - C)$$

We construct the Lagrangian function as follows:

$$\mathcal{L}(x,y,z,\lambda) = \frac{x(11x+10z+50) + y(11y+10z+50) + z(10x+10y+21z+10)}{2} - \lambda(x+y+z-2) \tag{6}$$

The partial derivatives of the function $\mathcal{L}$ are computed as follows:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= \frac{1}{2}[x(11x+10z+50) + y(11y+10z+50) + z(10x+10y+21z+10)] - \lambda(x+y+z-2) \\
&= \frac{1}{2}(11x^2 + 10zx + 50x + 11y^2 + 10zy + 50y + 10xz + 10yz + 21z^2 + 10z) - \lambda x - \lambda y - \lambda z + 2\lambda \\
&= \frac{1}{2}(22x + 10z + 50 + \cancel{11y^2} + \cancel{10zy} + \cancel{50y} + 10z + \cancel{10yz} + \cancel{21z^2} + \cancel{10z}) - \lambda - \cancel{\lambda y} - \cancel{\lambda z} + \cancel{2\lambda} \\
&= \frac{1}{2}(22x + 10z + 50 + 10z) - \lambda \\
&= \frac{1}{2}(22x + 20z + 50) - \lambda \\
&= 11x + 10z + 25 - \lambda
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial y} &= \frac{1}{2}[x(11x + 10z + 50) + y(11y + 10z + 50) + z(10x + 10y + 21z + 10)] - \lambda(x + y + z - 2) \\
&= \frac{1}{2}(11x^2 + 10zx + 50x + 11y^2 + 10zy + 50y + 10xz + 10yz + 21z^2 + 10z) - \lambda x - \lambda y - \lambda z + 2\lambda \\
&= \frac{1}{2}(\cancel{11x^2} + \cancel{10zx} + \cancel{50x} + 22y + 10z + 50 + \cancel{10xz} + 10z + \cancel{21z^2} + \cancel{10z}) - \cancel{\lambda x} - \lambda - \cancel{\lambda z} + \cancel{2\lambda} \\
&= \frac{1}{2}(22y + 10z + 50 + 10z) - \lambda \\
&= \frac{1}{2}(22y + 20z + 50) - \lambda \\
&= 11y + 10z + 25 - \lambda
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial z} &= \frac{1}{2}[x(11x + 10z + 50) + y(11y + 10z + 50) + z(10x + 10y + 21z + 10)] - \lambda(x + y + z - 2) \\
&= \frac{1}{2}(11x^2 + 10zx + 50x + 11y^2 + 10zy + 50y + 10xz + 10yz + 21z^2 + 10z) - \lambda x - \lambda y - \lambda z + 2\lambda \\
&= \frac{1}{2}(\cancel{11x^2} + 10x + \cancel{50x} + \cancel{11y^2} + 10y + \cancel{50y} + 10x + 10y + 42z + 10) - \cancel{\lambda x} - \cancel{\lambda y} - \lambda + \cancel{2\lambda} \\
&= \frac{1}{2}(10x + 10y + 10x + 10y + 42z + 10) - \lambda \\
&= \frac{1}{2}(20x + 20y + 42z + 10) - \lambda \\
&= 21z + 10x + 10y + 5 - \lambda
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \lambda} &= \cancel{\frac{1}{2}}[\cancel{x(11x + 10z + 50)} + \cancel{y(11y + 10z + 50)} + \cancel{z(10x + 10y + 21z + 10)}] - \lambda(x + y + z - 2) \\
&= -\lambda x - \lambda y - \lambda z + 2\lambda \\
&= -x - y - z + 2 \\
&= 2 - x - y - z
\end{aligned}
$$

Now, we solve the following system of equations as we set $\nabla \mathcal{L} = \vec{0}$:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= 11x + 10z + 25 - \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial y} &= 11y + 10z + 25 - \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial z} &= 21z + 10x + 10y + 5 - \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= 2 - x - y - z = 0
\end{aligned}
$$

We use Cramer's Rule to solve the system of equations. First, we rearrange the system:

$$
\begin{aligned}
11x + 10z - \lambda &= -25 \\
11y + 10z - \lambda &= -25 \\
21z + 10x + 10y - \lambda &= -5 \\
-x - z - y &= -2
\end{aligned}
$$

Next, the determinant of the coefficient matrix[6]:

$$D = \begin{vmatrix} 11 & 10 & 0 & -1 \\ 0 & 10 & 11 & -1 \\ 10 & 21 & 10 & -1 \\ -1 & -1 & -1 & 0 \end{vmatrix}$$

The answer column:

$$\begin{vmatrix} -25 \\ -25 \\ -5 \\ -2 \end{vmatrix}$$

Evaluate the determinant of the coefficient matrix:

$$\begin{aligned}
D &= \begin{vmatrix} 11 & 10 & 0 & -1 \\ 0 & 10 & 11 & -1 \\ 10 & 21 & 10 & -1 \\ -1 & -1 & -1 & 0 \end{vmatrix} \\
&= 11 \begin{vmatrix} \cancel{11} & \cancel{10} & \cancel{0} & \cancel{-1} \\ \cancel{0} & 10 & 11 & -1 \\ \cancel{10} & 21 & 10 & -1 \\ \cancel{-1} & -1 & -1 & 0 \end{vmatrix} - 10 \begin{vmatrix} \cancel{11} & \cancel{10} & \cancel{0} & \cancel{-1} \\ 0 & \cancel{10} & 11 & -1 \\ 10 & \cancel{21} & 10 & -1 \\ -1 & \cancel{-1} & -1 & 0 \end{vmatrix} + 0 \begin{vmatrix} \cancel{11} & \cancel{10} & \cancel{0} & \cancel{-1} \\ 0 & 10 & \cancel{11} & -1 \\ 10 & 21 & \cancel{10} & -1 \\ -1 & -1 & \cancel{-1} & 0 \end{vmatrix} - (-1) \begin{vmatrix} \cancel{11} & \cancel{10} & \cancel{0} & \cancel{-1} \\ 0 & 10 & 11 & \cancel{-1} \\ 10 & 21 & 10 & \cancel{-1} \\ -1 & -1 & -1 & \cancel{0} \end{vmatrix} \\
&= 11 \begin{vmatrix} 10 & 11 & -1 \\ 21 & 10 & -1 \\ -1 & -1 & 0 \end{vmatrix} - 10 \begin{vmatrix} 0 & 11 & -1 \\ 10 & 10 & -1 \\ -1 & -1 & 0 \end{vmatrix} + 0 + 1 \begin{vmatrix} 0 & 10 & 11 \\ 10 & 21 & 10 \\ -1 & -1 & -1 \end{vmatrix} \\
&= 11[(10)(10)(0) + (11)(-1)(-1) + (-1)(21)(-1) - (-1)(10)(-1) - (-1)(-1)(10) - (0)(21)(11)] \\
&\quad - 10[(0)(10)(0) + (11)(-1)(-1) + (-1)(10)(-1) - (-1)(10)(-1) - (-1)(-1)(0) - (0)(10)(11)] \\
&\quad + 1[(0)(21)(-1) + (10)(10)(-1) + (11)(10)(-1) - (-1)(21)(11) - (-1)(10)(0) - (-1)(10)(10)] \\
&= 11(12) - 10(11) + 1(121) \\
&= 143
\end{aligned}$$

Evaluate the determinant of the coefficient matrix with the answer column replacing the x-column[7]:

$$\begin{aligned} D_x &= \begin{vmatrix} -25 & 10 & 0 & -1 \\ -25 & 10 & 11 & -1 \\ -5 & 21 & 10 & -1 \\ -2 & -1 & -1 & 0 \end{vmatrix} \\ &= 22 \end{aligned}$$

Evaluate the determinant of the coefficient matrix with the answer column replacing the z-column:

$$\begin{aligned} D_z &= \begin{vmatrix} 11 & -25 & 0 & -1 \\ 0 & -25 & 11 & -1 \\ 10 & -5 & 10 & -1 \\ -1 & -2 & -1 & 0 \end{vmatrix} \\ &= 242 \end{aligned}$$

[6] The column order of the coefficients is $x, z, y,$ and λ

[7] We will not show the steps here. However, the steps should be the same as when we calculated the determinant of the coefficient matrix.

Evaluate the determinant of the coefficient matrix with the answer column replacing the y-column:

$$D_y = \begin{vmatrix} 11 & 10 & -25 & -1 \\ 0 & 10 & -25 & -1 \\ 10 & 21 & -5 & -1 \\ -1 & -1 & -2 & 0 \end{vmatrix}$$
$$= 22$$

Evaluate the determinant of the coefficient matrix with the answer column replacing the λ-column:

$$D_\lambda = \begin{vmatrix} 11 & 10 & 0 & -25 \\ 0 & 10 & 11 & -25 \\ 10 & 21 & 10 & -5 \\ -1 & -1 & -1 & -2 \end{vmatrix}$$
$$= 6237$$

Using Crammer's rule:

$$x = \frac{D_x}{D} = \frac{22}{143} \approx 0.1538$$
$$z = \frac{D_z}{D} = \frac{242}{143} \approx 1.6923$$
$$y = \frac{D_y}{D} = \frac{22}{143} \approx 0.1538$$
$$\lambda = \frac{D_\lambda}{D} = \frac{6237}{143} \approx 43.6154$$

Here, the the value of the Lagrange Multiplier, λ, represents the rate of change of the optimum value of $A(x, y, z)$ as C increases (where $g(x, y, z) = C$). In our context, it can be interpreted as the the rate of change of average travel time as the total flow, $|\Phi|$, increases. Thus, the minimum average travel time can be computed as follows:

$$A(\frac{22}{143}, \frac{22}{143}, \frac{242}{143}) = \frac{\frac{22}{143}(11(\frac{22}{143}) + 10(\frac{242}{143}) + 50) + (\frac{22}{143})(11(\frac{22}{143}) + 10(\frac{242}{143}) + 50) + (\frac{242}{143})(10(\frac{22}{143}) + 10(\frac{22}{143}) + 21(\frac{242}{143}) + 10)}{2}$$
$$\approx 51.6923$$

7.3 Second Derivative Test

f) Eliminating the z variable by substituting $z = 2 - x - y$ into $A(x, y, z)$:

$$\begin{aligned}
A(x,y) &= \frac{x(11x + 10(2 - x - y) + 50) + y(11y + 10(2 - x - y) + 50) + (2 - x - y)(10x + 10y + 21(2 - x - y) + 10)}{2} \\
&= \frac{x(11x + 20 - 10x - 10y + 50) + y(11y + 20 - 10x - 10y + 50) + (2 - x - y)(10x + 10y + 42 - 21x - 21y + 10)}{2} \\
&= \frac{x(x - 10y + 70) + y(y - 10x + 70) + (2 - x - y)(-11x - 11y + 52)}{2} \\
&= \frac{x^2 - 10yx + 70x + y^2 - 10xy + 70y + (-22x - 22y + 104 + 11x^2 + 11xy - 52x + 11xy + 11y^2 - 52y)}{2} \\
&= \frac{(x^2 + 11x^2) + (y^2 + 11y^2) + (70x - 22x - 52x) + (70y - 22y - 52y) - (10yx + 10xy - 11xy - 11xy) + 104}{2} \\
&= \frac{12x^2 + 12y^2 - 4x - 4y - (-2xy) + 104}{2} \\
&= \frac{12x^2 + 12y^2 - 4x - 4y + 2xy + 104}{2} \\
&= \frac{\cancel{2}(6x^2 + 6y^2 - 2x - 2y + xy + 52)}{\cancel{2}} \\
&= 6x^2 + 6y^2 - 2x - 2y + xy + 52
\end{aligned}$$

Computing the first-order partial derivatives of the function A and setting them equal to zero:

$$\frac{\partial A}{\partial x} = 12x - 2 + y = 0$$

$$\frac{\partial A}{\partial y} = 12y - 2 + x = 0$$

We find the following critical point:

$$\begin{aligned}
12x - 2 + y &= 0 \\
y &= -12x + 2
\end{aligned}$$

Therefore:

$$\begin{aligned}
12(-12x + 2) - 2 + x &= 0 \\
-144x + 24 - 2 + x &= 0 \\
-143x + 22 &= 0 \\
-143x &= -22 \\
x &= \frac{-22}{-143} \\
x &= \frac{22}{143} \approx 0.1538
\end{aligned}$$

And:

$$\begin{aligned}
y &= -12x + 2 \\
y &= -12(\frac{22}{143}) + 2 \\
y &= (\frac{-264}{143}) + \frac{286}{143} \\
y &= (\frac{-264 + 286}{143}) \\
y &= \frac{22}{143} \approx 0.1538
\end{aligned}$$

As can be seen, the point $(\frac{22}{143}, \frac{22}{143})$ is a critical point where $\nabla A(\frac{22}{143}, \frac{22}{143}) = \vec{0}$. To conduct the second derivative test, we compute the second-order partial derivatives:

$$\frac{\partial^2 A}{\partial x^2} = 12 \qquad \frac{\partial^2 A}{\partial x \partial y} = 1$$
$$\frac{\partial^2 A}{\partial y \partial x} = 1 \qquad \frac{\partial^2 A}{\partial y^2} = 12$$

Next, we find the discriminant:

$$\begin{aligned} D &= A_{xx}(\frac{22}{143}, \frac{22}{143})A_{yy}(\frac{22}{143}, \frac{22}{143}) - (A_{xy}(\frac{22}{143}, \frac{22}{143}))^2 \\ &= 12 \cdot 12 - (1)^2 \\ &= 143 \end{aligned}$$

The **Second Derivative Test** states that:

- If $D = 143 > 0$ and $A_{xx}(\frac{22}{143}, \frac{22}{143}) = 12 > 0$, then $A(x, y)$ has a local minimum at the point $(\frac{22}{143}, \frac{22}{143})$.

We confirm that the minimum average travel time ≈ 51.6923 computed in part e) is, indeed, a minimum.

7.4 Stability of minimum average travel time

g) When all drivers are distributed according to the optimal arrangement:

$$x = \Phi_{abz} = \frac{22}{143} \qquad z = \Phi_{abcz} = \frac{242}{143} \qquad y = \Phi_{acz} = \frac{22}{143}$$

$T(\Phi_{abz})$ and $T(\Phi_{acz})$ can be computed as follows:

$$\begin{aligned} t_1(\frac{22}{143} + \frac{242}{143}) &= 10 \times (\frac{22}{143} + \frac{242}{143}) \approx 18.4615 \\ t_2(\frac{22}{143}) &= 50 + \frac{22}{143} \approx 50.1538 \\ t_3(\frac{22}{143} + \frac{242}{143}) &= 10 \times (\frac{22}{143} + \frac{242}{143}) \approx 18.4615 \\ t_4(\frac{22}{143}) &= 50 + \frac{22}{143} \approx 50.1538 \end{aligned}$$

Therefore:

$$\begin{aligned} T(\Phi_{abz}) &= t_1(\frac{22}{143} + \frac{242}{143}) + t_4(\frac{22}{143}) \\ &= 18.4615 + 50.1538 \\ &\approx 68.6153 \end{aligned}$$

$$\begin{aligned} T(\Phi_{acz}) &= t_2(\frac{22}{143}) + t_3(\frac{22}{143} + \frac{242}{143}) \\ &= 50.1538 + 18.4615 \\ &\approx 68.6153 \end{aligned}$$

$T(\Phi_{abcz})$ can be computed as follows:

$$\begin{aligned} t_1(\frac{22}{143} + \frac{242}{143}) &= 10 \times (\frac{22}{143} + \frac{242}{143}) \approx 18.4615 \\ t_3(\frac{22}{143} + \frac{242}{143}) &= 10 \times (\frac{22}{143} + \frac{242}{143}) \approx 18.4615 \\ t_5(\frac{242}{143}) &= 10 + \frac{242}{143} \approx 11.6923 \end{aligned}$$

Therefore:

$$\begin{aligned}
T(\Phi_{abcz}) &= t_1(\frac{22}{143}+\frac{242}{143})+t_5(\frac{242}{143})+t_3(\frac{22}{143}+\frac{242}{143})\\
&= 18.4615+11.6923+18.4615\\
&\approx 48.6153
\end{aligned}$$

The travel times are as follows:

$$\begin{aligned}
T(\Phi_{abz}) \ \ and \ \ T(\Phi_{acz}) &\approx 68.6153\\
T(\Phi_{abcz}) &\approx 48.6153
\end{aligned}$$

As can be seen, travel time on path **abcz** is shorter than those of the other two paths. There will be an incentive for drivers traveling on paths **abz** and **acz** to switch to path **abcz**. The minimum average travel time that we observe is not stable. We will examine an intermediary scenario where $\frac{1}{10}$ of the drivers on path **abz** switch to path **abcz**. The new arrangement is as follows:

$$z=\Phi_{abcz}=\frac{242}{143}+\frac{1}{10}=\frac{233}{130},\quad x=\Phi_{abz}=\frac{22}{143}-\frac{1}{10}=\frac{7}{130},\quad y=\Phi_{acz}=\frac{22}{143}$$

The new average travel time can be computed as follows:

$$\begin{aligned}
A(x,y,z) &= \frac{x(11x+10z+50)+y(11y+10z+50)+z(10x+10y+21z+10)}{2}\\
&= \frac{(\frac{7}{130})(11(\frac{7}{130})+10(\frac{233}{130})+50)+(\frac{22}{143})(11(\frac{22}{143})+10(\frac{233}{130})+50)+(\frac{233}{130})(10(\frac{7}{130})+10(\frac{22}{143})+21(\frac{233}{130})+10)}{2}\\
&\approx 51.7523
\end{aligned}$$

As can be seen, the new average time $\approx 51.7523 > 51.6923 \approx$ minimum average time. The new travel times can be computed as follows:

$$\begin{aligned}
t_1(\frac{7}{130}+\frac{233}{130}) &= 10\times(\frac{7}{130}+\frac{233}{130})\approx 18.4615\\
t_2(\frac{22}{143}) &= 50+\frac{22}{143}\approx 50.1538\\
t_3(\frac{22}{143}+\frac{233}{130}) &= 10\times(\frac{22}{143}+\frac{233}{130})\approx 19.4615\\
t_4(\frac{7}{130}) &= 50+\frac{7}{130}\approx 50.0538
\end{aligned}$$

Therefore:

$$\begin{aligned}
T(\Phi_{abz}) &= t_1(\frac{7}{130}+\frac{233}{130})+t_4(\frac{7}{130})\\
&= 18.4615+50.0538\\
&= 68.5153
\end{aligned}$$

$$\begin{aligned}
T(\Phi_{acz}) &= t_2(\frac{22}{143})+t_3(\frac{22}{143}+\frac{233}{130})\\
&= 50.1538+19.4615\\
&= 69.6153
\end{aligned}$$

$T(\Phi_{abcz})$ can be computed as follows:

$$\begin{aligned}
t_1(\frac{7}{130}+\frac{233}{130}) &= 10\times(\frac{7}{130}+\frac{233}{130})\approx 18.4615\\
t_3(\frac{22}{143}+\frac{233}{130}) &= 10\times(\frac{22}{143}+\frac{233}{130})\approx 19.4615\\
t_5(\frac{233}{130}) &= 10+\frac{233}{130}\approx 11.7923
\end{aligned}$$

Therefore:

$$
\begin{aligned}
T(\Phi_{abcz}) &= t_1\left(\frac{7}{130} + \frac{233}{130}\right) + t_5\left(\frac{233}{130}\right) + t_3\left(\frac{22}{143} + \frac{233}{130}\right) \\
&= 18.4615 + 11.7923 + 19.4615 \\
&= 49.7153
\end{aligned}
$$

The new travel times are as follows:

$$
\begin{aligned}
T(\Phi_{abz}) &\approx 68.5153 \\
T(\Phi_{acz}) &\approx 69.6153 \\
T(\Phi_{abcz}) &\approx 49.7153
\end{aligned}
$$

Travel time on path **abcz** is still shorter. Drivers on paths **abz** and **acz** will re-route to path **abcz** until the distribution is once again following the optimal arrangement:

$$z = \Phi_{abcz} = 2, \quad x = y = \Phi_{abz} = \Phi_{acz} = 0, \quad T(\Phi) = 52$$

The new average travel time can be computed as follows:

$$
\begin{aligned}
A(x, y, z) &= \frac{x(11x + 10z + 50) + y(11y + 10z + 50) + z(10x + 10y + 21z + 10)}{2} \\
&= \frac{\cancel{0(11(0) + 10(2) + 50)} + \cancel{(0)(11(0) + 10(2) + 50)} + (2)(\cancel{10(0)} + \cancel{10(0)} + 21(2) + 10)}{2} \\
&= \frac{2(42 + 10)}{2} \\
&= \frac{104}{2} \\
&= 52
\end{aligned}
$$

Notice that the new average travel time $A_{optimal}(0, 0, 2) = 52 > 51.6923 = A_{minimum}(\frac{22}{143}, \frac{22}{143}, \frac{242}{143})$. One could argue that results like these make a great case for autonomous vehicles with a centralized navigation center that optimizes the minimum average commute time for the entire group rather than individual drivers all acting in self-interest.

7.5 Removing link 5 and the presence of Braess' paradox in case a) from Task I & II

h) Removing link 5 leaves two paths **abz** and **acz**; this essentially eliminates the z variable. The new objective function is:

$$
\begin{aligned}
A(x, y) &= \frac{x \cdot T_{abz}(x, y) + y \cdot T_{acz}(x, y)}{2} \\
&= \frac{x \cdot (10x + (50 + x)) + y \cdot ((50 + y) + 10y)}{2} \\
&= \frac{x \cdot (11x + 50) + y \cdot (11y + 50)}{2}
\end{aligned}
$$

and it is subject to the constraint:

$$g(x, y) = x + y = 2 \tag{7}$$

We construct the Lagrangian function as follows:

$$\mathcal{L}(x, y, \lambda) = \frac{x(11x + 50) + y(11y + 50)}{2} - \lambda(x + y - 2) \tag{8}$$

The partial derivatives of the function $\mathcal{L}$ are computed as follows:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= \frac{1}{2}[x(11x+50)+y(11y+50)] - \lambda(x+y-2)\\
&= \frac{1}{2}(11x^2+50x+11y^2+50y) - \lambda x - \lambda y + 2\lambda\\
&= \frac{1}{2}(22x+50+\cancel{11y^2}+\cancel{50y}) - \lambda - \cancel{\lambda y} + \cancel{2\lambda}\\
&= \frac{1}{2}(22x+50) - \lambda\\
&= 11x+25-\lambda
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial y} &= \frac{1}{2}[x(11x+50)+y(11y+50)] - \lambda(x+y-2)\\
&= \frac{1}{2}(11x^2+50x+11y^2+50y) - \lambda x - \lambda y + 2\lambda\\
&= \frac{1}{2}(\cancel{11x^2}+\cancel{50x}+22y+50) - \cancel{\lambda x} - \lambda + \cancel{2\lambda}\\
&= \frac{1}{2}(22y+50) - \lambda\\
&= 11y+25-\lambda
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \lambda} &= \cancel{\frac{1}{2}}[\cancel{x(11x+50)}+\cancel{y(11y+50)}] - \lambda(x+y-2)\\
&= -\lambda x - \lambda y + 2\lambda\\
&= -x-y+2\\
&= 2-x-y
\end{aligned}$$

Now, we solve the following system of equations as we set $\nabla \mathcal{L} = \vec{0}$:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= 11x+25-\lambda = 0\\
\frac{\partial \mathcal{L}}{\partial y} &= 11y+25-\lambda = 0\\
\frac{\partial \mathcal{L}}{\partial \lambda} &= 2-x-y = 0
\end{aligned}$$

Since we eliminated the z variable, we may find it easier to solve the system of equations using substitution. First, we isolate for x:

$$\begin{aligned}
11x+25-\lambda &= 0\\
11x &= \lambda - 25\\
x &= \frac{\lambda-25}{11}
\end{aligned}$$

Isolate for y:

$$\begin{aligned}
11y+25-\lambda &= 0\\
11y &= \lambda - 25\\
y &= \frac{\lambda-25}{11}
\end{aligned}$$

Substitute the expressions for x and y into the third equation:

$$\begin{aligned}
2 - x - y &= 0 \\
2 - \left(\frac{\lambda - 25}{11}\right) - \left(\frac{\lambda - 25}{11}\right) &= 0 \\
2 - 2\left(\frac{\lambda - 25}{11}\right) &= 0
\end{aligned}$$

Isolate for λ:

$$\begin{aligned}
2 - 2\left(\frac{\lambda - 25}{11}\right) &= 0 \\
-2\left(\frac{\lambda - 25}{11}\right) &= -2 \\
\frac{\lambda - 25}{11} &= \frac{-2}{-2} \\
\lambda - 25 &= 1 \cdot 11 \\
\lambda &= 11 + 25 \\
\lambda &= 36
\end{aligned}$$

Substitute λ into the expressions for x and y:

$$\begin{aligned}
x &= \frac{\lambda - 25}{11} & y &= \frac{\lambda - 25}{11} \\
x &= \frac{36 - 25}{11} & y &= \frac{36 - 25}{11} \\
x &= 1 & y &= 1
\end{aligned}$$

The solutions are:

$$\begin{aligned}
x &= 1 \\
y &= 1 \\
\lambda &= 36
\end{aligned}$$

The minimum average travel time can be computed as follows:

$$\begin{aligned}
A(1,1) &= \frac{(1) \cdot (11(1) + 50) + (1) \cdot (11(1) + 50)}{2} \\
&= 61
\end{aligned}$$

To perform the Second Derivative Test, we compute the first-order partial derivatives of the function A and setting them equal to zero:

$$\begin{aligned}
\frac{\partial A}{\partial x} &= \frac{1}{2}(11x^2 + 50x + 11y^2 + 50y) \\
&= \frac{1}{2}(22x + 50 + \cancel{11y^2} + \cancel{50y}) \\
&= \frac{1}{2}(22x + 50) \\
&= 11x + 25 \\
\frac{\partial A}{\partial x} &= 11x + 25 = 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial A}{\partial y} &= \frac{1}{2}(11x^2 + 50x + 11y^2 + 50y) \\
&= \frac{1}{2}(\cancel{11x^2} + \cancel{50x} + 22y + 50) \\
&= \frac{1}{2}(22y + 50) \\
&= 11y + 25 \\
\frac{\partial A}{\partial y} &= 11y + 25 = 0
\end{aligned}$$

We find the following critical point:

$$\begin{aligned}
11x + 25 &= 0 \\
11x &= -25 \\
x &= \frac{-25}{11}
\end{aligned}$$

$$\begin{aligned}
11y + 25 &= 0 \\
11y &= -25 \\
y &= \frac{-25}{11}
\end{aligned}$$

We compute the second-order partial derivatives:

$$\frac{\partial^2 A}{\partial x^2} = 11 \qquad \frac{\partial^2 A}{\partial x \partial y} = 0$$

$$\frac{\partial^2 A}{\partial y \partial x} = 0 \qquad \frac{\partial^2 A}{\partial y^2} = 11$$

Next, we find the discriminant:

$$\begin{aligned}
D &= A_{xx}(\frac{-25}{11}, \frac{-25}{11})A_{yy}(\frac{-25}{11}, \frac{-25}{11}) - (A_{xy}(\frac{-25}{11}, \frac{-25}{11}))^2 \\
&= 11 \cdot 11 - (0)^2 \\
&= 121
\end{aligned}$$

The **Second Derivative Test** states that:

- If $D = 121 > 0$ and $A_{xx}(\frac{-25}{11}, \frac{-25}{11}) = 11 > 0$, then $A(x, y)$ has a local minimum at the point $(\frac{-25}{11}, \frac{-25}{11})$.

We confirm that the new minimum average travel time = 61 is a minimum. Removing link 5 leads to a longer minimum average travel time: $A_{\text{no link 5}}(1, 1) = 61 > 51.6923 = A_{minimum}(\frac{22}{143}, \frac{22}{143}, \frac{242}{143})$.

i) Braess' paradox is a phenomenon in which the removal of an edge in a transportation network results in improved flow. In part e), we found that the minimum average travel time is ≈ 51.6923 before removing a link. Removing link 5 actually worsens traffic flow as the new minimum average travel time is 61, which we computed in part h). Therefore, Braess' paradox is not present in case (a) regardless of whether maximum travel time(as we have verified in Task V) or average travel time is used. For both measures, removing link 5 does not result in improved traffic.

8 Task VIII

8.1 Real-world examples of Braess' paradox

a) An in-depth search for real-world examples of Braess' paradox yields the following list:

- In Seoul, South Korea, a speeding up of traffic around the city was seen when a motorway was removed as part of the Cheonggyecheon restoration project. Traffic improved when "a typically traffic-choked elevated highway was taken down and turned into a waterway, and public recreation area..."

Figure 4: https://www.landscapeperformance.org/case-study-briefs/cheonggyecheon-stream-restoration

In a Cornell University course blog, the author explains the seemingly counter-intuitive phenomenon: "[i]n the presence of the highway (with 'optimum' drive time), all drivers would prefer to use the road, thus congesting it. However, in its absence, this traffic would get scattered across several other roads, which would reduce total travel time." The author does not specify which measure of travel time was used by the Korean officials to assess the quality of traffic flow and how much traffic "sped up": it would be interesting to apply the measures we use in this project to see if conclusions would change. The blog, however, notes that the observed improvements in traffic flow cannot be attributed to Braess' paradox solely. "Factors such as the government's increased investments in public transit systems would also have played a part in this." Moreover, as we have discussed in Task VI, the author notes the assumption that all drivers are rational agents may not be realistic. "[O]n a particular day, more drivers may just happen to choose one route over the other, changing optimum travel routes and times." We include the link to the blog in our reference page.

- In 2008, the State of Massachusetts invested $15 billion in a transportation project. The so-called *Big Dig* "essentially increases the capacity of the central highway and its interchanges by placing the entire system as a series of tunnels under the central business district." The following diagram shows a road map of the area in which the tunnels were constructed:

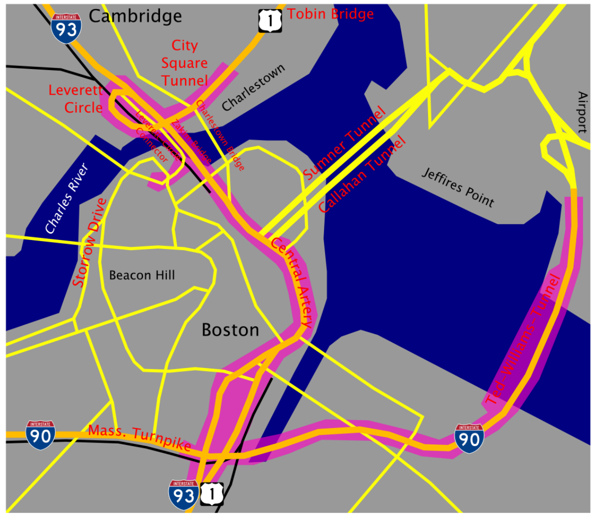

Figure 5: https://upload.wikimedia.org/wikipedia/commons/7/78/Boston-big-dig-area.png

While the Big Dig relieved traffic on highways in the downtown area, "the bottlenecks were only pushed outward, as more drivers jockey for the limited space on the major commuting routes." In the article we found, the writer reports the following case to demonstrate that the addition of the tunnels had, in fact, worsened traffic on some roads:

> "The worst increase has been along I-93 northbound during the evening commute. In 1994, before the tunnels were dug, it took, on average, 12 minutes at peak evening rush hour to go the 11 miles from the Zakim Bridge to the Route 128 interchange in Woburn. Now it takes 25 minutes, double the time."

The article notes that a possible cause of the increased travel time is that the Big Dig solved the worst bottleneck that is the financial district at the cost of increasing congestion elsewhere. Frederick P. Salvucci, the former state transportation secretary who pushed the project in the 1970s and 1980s, echoed this idea by claiming that "he and others anticipated that there would be bigger delays outside the city..." In a Cornell University course blog, the author notes that Braess' paradox may have played a role in the scenario here. That is, the financial district is not the only common destination. So, by adding new alternative routes that drivers may switch to, the new transportation system freed up roads in the financial district but, as an unfortunately side-effect, increased traffic elsewhere. An interesting idea to ponder here is that the cause of increased travel time may be as simple as this— there are more cars and trucks today than there were in 1994. Perhaps, all of the above played a role in the worsening of traffic in Boston. Again, we include the link to the blog and the article in our reference page.

- In **Youn, Jeong, Gastner (2008)**, the authors examine three transportation networks: Boston-Cambridge area, London, UK, and New York City. The diagram on the following page illustrates the three networks.

  In the paper, the authors use a concept called the price of anarchy (PoA) to demonstrate that, in some instances, the closing of several links in a transportation network results in improved traffic flow. The price of anarchy, intuitively speaking, measures the degree to which the "efficiency of a system degrades due to selfish behavior of its agents." In some context, it can be thought of as the "efficiency-loss ratio." Greek theoretical computer scientist Christos Papadimitriou has described the efficiency-loss ratio, in the context of game theory, as a measure that quantifies the effect of a system's "lack of coordination." In **Youn, Jeong and Gastner (2008)**, the author reports the following findings:

  > "In most cases, the [PoA] increases when one street is blocked, as intuitively expected. Nonetheless, there are six connections which, if one is removed, decrease the delay in the

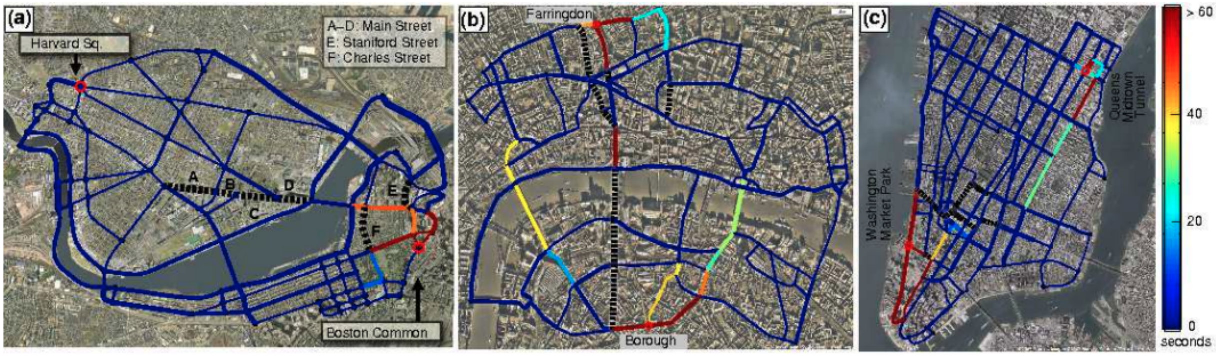

Figure 6: https://arxiv.org/pdf/0712.1598v4.pdf

> Nash equilibrium, shown as dotted lines in [Figure 6]. If all drivers ideally cooperated to reach the social optimum, these roads could be helpful; otherwise it is better to close these streets."

In the same paper, the authors find empirical evidence of "seven links causing Braess's paradox in London (F=10,000) and twelve in New York (F=18,000)..." They argue that the existence of such links "under the investigated conditions suggests that Braess's paradox is more than an academic curiosity or an anecdote with only sketchy empirical evidence." Here, the authors' approach to analyzing transportation networks— namely, their using the concept of PoA instead of some measure of travel time to investigate whether Braess' paradox is present— led them to a different conclusion than that of **Steinberg and Zangwill (1983)**, which states: "[t]he present paper gives, under reasonable assumptions, . . . that Braess' Paradox is about as likely to occur as not occur." This difference at least underscores the notion that different approaches to analyzing traffic systems can lead to varying conclusions.

8.2 Is Nassau street the one to blame for bad traffic?

b) We observe a possible example of Braess' paradox in the State of New Jersey. The city of Princeton is home of the prestigious Princeton University, which receives a great deal of visitors on a daily basis. To the north of Princeton's beautiful campus is Nassau street, one of the most distinguished streets in Colonial American history and is the heart of Princeton. Many of the town's famous restaurants, bars, and shops call this street home:

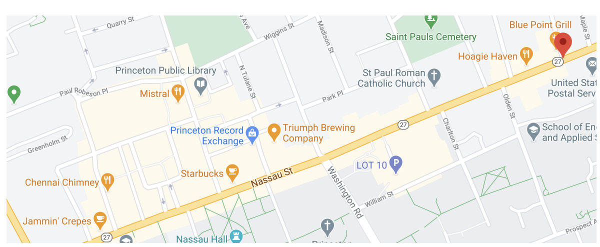

Figure 7: Nassau Street

In addition to being a fan-favorite, Nassau street is notorious in the eyes of local residents for its traffic. Above all, the street has many intersections and traffic lights. In normal times, due to the relatively high volume of visitors, traffic can be very slow. As a result, many drivers on Nassau street re-route as soon as they see the opportunity. Their behaviors often lead to more congestion on intersecting roads (which connect local residential drive-ways, elementary school zones, etc.) beyond the dining and shopping blocks. This problem is compounded by the fact that *most* roads, including Nassau, are two-way roads, meaning there's traffic flowing towards and away from Nassau street at all times. In many cases, drivers would re-route to avoid taking Nassau completely. Below, we attach an annotated google map image to illustrate what we mean.

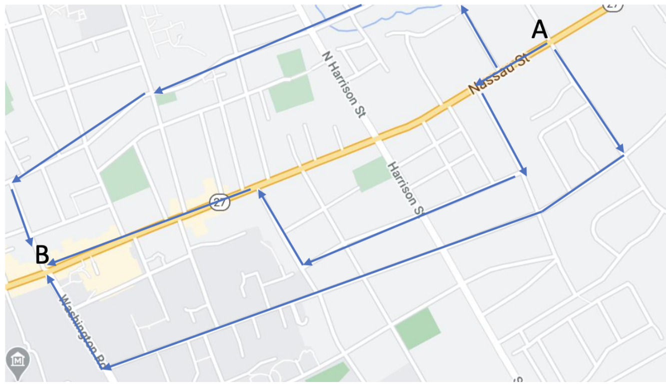

Figure 8: Re-route

As can be seen, for an origin-destination pair **AB**, there exists many different edges that drivers may choose to take. In Figure 8, we highlight a few of them in blue. Keep in mind that the arrows simply indicate the direction of drivers who seek to go from node A to node B. Most of these intersecting side-roads are two-way roads, so there are drivers traveling both ways. On any given day, whenever traffic on these different paths is noticeably different, drivers will have an incentive to switch. Intuition may lead us to believe that having more alternative routes may help relieve traffic on Nassau. That is, more drivers would head off Nassau in pursuit of shorter travel times, leaving fewer drivers and thus congestion on Nassau. However, the reality is rather more complicated. The problem is that Nassau's bad traffic is having some *spillover effect* on the roads and blocks surrounding it. Blocks and roads near Nassau that wouldn't have otherwise experienced sub-optimal traffic are all of a sudden seeing more congestion as a result of drivers on Nassau acting in self-interest. We may consider again using the concept of average travel time. Here, redistribution of drivers onto roads and blocks surrounding Nassau has led to higher average travel times for the whole group of drivers in this area. In this situation, we argue that removing some of these edges in Figure 8 by reducing some of the two-way roads to one-ways may improve overall traffic flow of the area. The idea is not a new one. In fact, a street called Riverside Dr, where a local elementary school is situated, becomes a one-way street everyday from 12pm to 3:30pm to prevent drivers on Nassau street from entering into the road (for reasons including safety of school children, parents pick-up parking space, and ease of school bus departures).

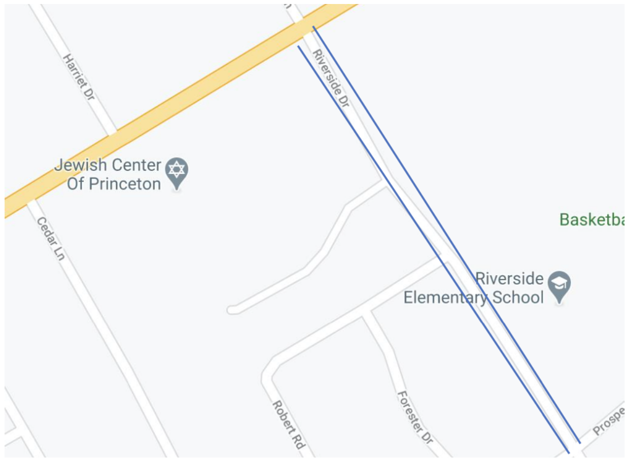

Figure 9: River Dr

In another example, the Princeton police department issued a message in June of 2020 titled "NEW traffic pattern: Witherspoon Street one-way only between Nassau Street and Spring Street."[8] The idea of reducing the number of two-way roads (essentially removing edges that drivers could use to re-route) to improve overall traffic flow in the area can be thought of as a potential real-world example of Braess' paradox. As to why removing edges may help, there is much debate. The underlying reasoning, as one neighbor once succinctly put, is this: "If bad traffic's got to hit somewhere, let it be Nassau. And you [*explicit*] sure hope it's Nassau."

[8] https://local.nixle.com/alert/8061060/

9 References

- Dietrich Braess. On a paradox of traffic planning. Transportation Science, 39(4):446–450, November 2005. Translation of the original 1968 paper "Uber ein Paradoxon aus der Verkehrsplanung" into English by Anna Nagurney and Tina Wakolbinger

- Kenneth M Monks, "Braess' Paradox in City Planning: An Application of Multivariable Optimization" 8 MAA Convergence (May 2020)

- S. Robinson, "The price of anarchy," SIAM News, vol. 37, no. 5, pp. 1–4, Jun. 2004.

- Youn, H., Gastner, M. T. Jeong, H. Price of anarchy in transportation networks: efficiency and optimality control. Phys Rev Lett 101, 128701 (2008).

- https://blogs.cornell.edu/info2040/2017/09/14/braess-paradox-the-cheonggyecheon-restoration-project/

- https://blogs.cornell.edu/info2040/2011/09/25/real-life-example-of-braesss-paradox-bostons-big-dig/#:~:text=This%20is%20called%20Braess%27s%20paradox,to%20date%3A%20The%20Big%20Dig.&text=While%20the%20Big%20Dig%20did,actually%20longer%20as%20a%20result

- http://archive.boston.com/news/local/articles/2008/11/16/big_dig_pushes_bottlenecks_outward/

- https://en.wikipedia.org/wiki/Price_of_anarchy